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Research Article

Exponential Stability of Linear Discrete Systems with Multiple Delays

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The paper investigates the exponential stability and exponential estimate of the norms of solutions to a linear system of difference equations with multiple delays $x(k+1) = Ax(k) + \sum_{i=1}^{s} B_i x(k - m_i)$, $k = 0, 1, \ldots$, where $s \in \mathbb{N}$, A and B_i are square matrices, and $m_i \in \mathbb{N}$. New criterion for exponential stability is proved by the Lyapunov method. An estimate of the norm of solutions is given as well and relations to the well-known results are discussed.

1. Preliminaries

The investigation of the stability of linear difference systems with delay is a constant priority of research. We refer, for example, to [1–14] and to the references therein.

The paper considers the exponential stability of linear discrete systems with multiple delays

$$x(k+1) = Ax(k) + \sum_{i=1}^{s} B_i x(k - m_i), \quad k = 0, 1, \ldots \tag{1}$$

where $s \in \mathbb{N}$, A and B_i are $n \times n$ matrices, and $m_i \in \mathbb{N}$. For (1) exponential-type stability and exponential estimate of the rate of convergence of solutions are derived.

Set $m := \max\{m_1, \ldots, m_s\}$. The initial Cauchy problem for system (1) is as follows:

$$x(k) = x_k \in \mathbb{R}, \quad k = -m, -m+1, \ldots. \tag{2}$$

For a vector $x = (x_1, \ldots, x_n)^T$, we define $|x|^2 := \sum_{i=1}^{n} x_i^2$. Let $\rho(A)$ be the spectral radius of the matrix A. Denote by $\lambda_{\max}(\mathscr{A})$ and $\lambda_{\min}(\mathscr{A})$ the maximum and the minimum eigenvalues, respectively, of a symmetric matrix $\mathscr{A}$ and define $\varphi(\mathscr{A}) := \lambda_{\max}(\mathscr{A})\lambda_{\min}^{-1}(\mathscr{A})$. For a given matrix $\mathscr{B}$, we use the norm defined by $|\mathscr{B}|^2 := \lambda_{\max}(\mathscr{B}^T\mathscr{B})$. In the paper, assume $|A| + \sum_{i=1}^{s} |B_i| > 0$.

The trivial solution $x(k) = 0$, $k = -m, -m+1, \ldots$ of (1) is called Lyapunov exponentially stable if there exist constants $N > 0$ and $\theta \in (0, 1)$ such that, for an arbitrary solution $x = x(k)$ of (1),

$$|x(k)| \le N \|x(0)\|_m \theta^k, \quad k = 1, 2, \ldots \tag{3}$$

where

$$\|x(0)\|_m := \max\{|x(i)|, \ i = -m, -m+1, \ldots, 0\}. \tag{4}$$

For the foundations of stability theory to difference equations, we refer, e.g., to [15, 16].

As it is customary, the asymptotic stability of (1) can be investigated by analyzing the roots of the related characteristic equation. The characteristic equation relevant to (1) is a polynomial equation of degree $(m+1)n$. For large m and n, it is impossible, in a general case, to solve such a problem. For example, the Schur-Cohn criterion [16, 17] is not applied because the computer calculation is too time-consuming.

Below, the exponential stability of (1) is analyzed by the second Lyapunov method and the following well-known result is utilized: if $\rho(A) < 1$, then the Lyapunov matrix equation

$$A^T HA - H = -C \tag{5}$$

has a unique solution, a positive definite symmetric matrix H for an arbitrary positive definite symmetric $n \times n$ matrix C (we refer, for example, to [16]).

In Section 2, the exponential stability of system (1) and exponential estimates of solutions are investigated. Concluding remarks and relations to the well-known results are included in Section 3.

2. Exponential Stability

Let $\gamma > 1$ be a parameter. Define auxiliary numbers

$$L_1 := \gamma \left[\lambda_{\max}(H) - \lambda_{\min}(C) + \sum_{i=1}^{s} \left| A^T H B_i \right| \right],$$
$$L_2 := \lambda_{\min}(H) - \frac{1}{2}\gamma\varphi(H) \left[2\sum_{i=1}^{s} \gamma^{m_i} \left| A^T H B_i \right| + \sum_{i,j=1}^{s} \left[\gamma^{m_i+1} + \gamma^{m_j+1} \right] \left| B_i^T H B_j \right| \right],$$
$$L_3 := \lambda_{\min}(C) - \sum_{i=1}^{s} \left| A^T H B_i \right| - \frac{\gamma-1}{\gamma}\lambda_{\max}(H) - \frac{1}{2} \cdot \varphi^2(H) \left[2\sum_{i=1}^{s} \gamma^{m_i} \left| A^T H B_i \right| + \sum_{i,j=1}^{s} \left[\gamma^{m_i+1} + \gamma^{m_j+1} \right] \left| B_i^T H B_j \right| \right]. \tag{6}$$

Theorem 1. *Let $\rho(A) < 1$, C be a fixed positive definite symmetric $n \times n$ matrix, let matrix H solve the equation (5), and, for a fixed $\gamma > 1$, let $L_1 > 0$, $L_2 > 0$, $L_3 \geq 0$. Then, system (1) is exponentially stable and, for an arbitrary solution $x = x(k)$, the estimate*

$$|x(k)| \leq \sqrt{\varphi(H)}\, \|x(0)\|_m \gamma^{-k/2}, \quad k \geq 1 \tag{7}$$

holds.

Proof. For the Lyapunov function $V(x,k) := \gamma^k x^T H x$, inequalities

$$\gamma^k \lambda_{\min}(H) |x|^2 \leq V(x,k) \leq \gamma^k \lambda_{\max}(H) |x|^2 \tag{8}$$

hold. Let $\delta := \varepsilon/\sqrt{\varphi(H)}$ where $\varepsilon > 0$ is given. Let a solution $x(k)$ of (1) satisfy $\|x(0)\|_m = \delta$. Then, for $k = -m, -m+1, \ldots, 0$,

$$\begin{aligned} V(x(k),k) &\leq \gamma^k \lambda_{\max}(H) |x(k)|^2 \\ &\leq \gamma^k \lambda_{\max}(H) \|x(0)\|_m^2 \leq \gamma^k \lambda_{\max}(H) \delta^2 \\ &= \gamma^k \lambda_{\max}(H) \frac{\varepsilon^2}{\varphi(H)} = \gamma^k \varepsilon^2 \lambda_{\min}(H) \\ &\leq \varepsilon^2 \lambda_{\min}(H), \end{aligned} \tag{9}$$

i.e.,

$$V(x(k),k) \leq \varepsilon^2 \lambda_{\min}(H). \tag{10}$$

Below, we prove that (10) is valid for $k = 1, 2, \ldots$, too. Assume, on the contrary, that (10) is not always valid. Then, an integer $k^* > 0$ exists such that, for $k = -m, -m+1, \ldots, k^*$, (10) holds, and, for $k = k^* + 1$,

$$V(x(k^*+1, k^*+1)) > \varepsilon^2 \lambda_{\min}(H). \tag{11}$$

Inequality (11) implies that, for $k = -m, -m+1, \ldots, k^*$,

$$\begin{aligned} \gamma^k \lambda_{\min}(H) |x(k)|^2 &\leq V(x(k),k) \leq \varepsilon^2 \lambda_{\min}(H) \\ &< V(x(k^*+1), k^*+1) \\ &\leq \gamma^{k^*+1} \lambda_{\max}(H) \left| x(k^*+1) \right|^2 \end{aligned} \tag{12}$$

and

$$|x(k)| < \gamma^{(k^*+1-k)/2} \sqrt{\varphi(H)} \left| x(k^*+1) \right|, \quad k = -m, -m+1, \ldots, k^*. \tag{13}$$

Now compute

$$\begin{aligned} \Delta V(x(k^*), k^*) &= V(x(k^*+1), k^*+1) - V(x(k^*), k^*) = \gamma^{k^*+1} x^T(k^*+1) H x(k^*+1) - \gamma^{k^*} x^T(k^*) H x(k^*) \\ &= \gamma^{k^*+1} \left[A x(k^*) + \sum_{i=1}^{s} B_i x(k^*-m_i) \right]^T \cdot H \left[A x(k^*) + \sum_{i=1}^{s} B_i x(k^*-m_i) \right] - \gamma^{k^*} x^T(k^*) \cdot H x(k^*). \end{aligned} \tag{14}$$

Rearranging this computation, we derive

$$\begin{aligned} \Delta V(x(k^*), k^*) &= -\gamma^{k^*+1} x^T(k^*) \left[H - A^T H A \right] x(k^*) \\ &+ 2\gamma^{k^*+1} x^T(k^*) A^T H \sum_{i=1}^{s} B_i x(k^*-m_i) \\ &+ \gamma^{k^*+1} \sum_{i,j=1}^{s} x^T(k^*-m_i) B_i^T H B_j x\left(k^*-m_j\right) \\ &+ \gamma^{k^*} (\gamma-1) x^T(k^*) H x(k^*). \end{aligned} \tag{15}$$

We estimate the first difference and use the assumption that the matrix H is a solution of equation (5); therefore,

$$\begin{aligned}\Delta V(x(k^*),k^*) &\le -\gamma^{k^*+1}\lambda_{\min}(C)|x(k^*)|^2 \\ &\quad + 2\gamma^{k^*+1}\sum_{i=1}^{s}\left|A^THB_i\right||x(k^*)||x(k^*-m_i)| \\ &\quad + \gamma^{k^*+1}\sum_{i,j=1}^{s}\left|B_i^THB_j\right||x(k^*-m_i)|\left|x\left(k^*-m_j\right)\right| \\ &\quad + \gamma^{k^*}(\gamma-1)\lambda_{\max}(H)|x(k^*)|^2\end{aligned} \tag{16}$$

and

$$\begin{aligned}\Delta V(x(k^*),k^*) &\le -\gamma^{k^*+1}\lambda_{\min}(C)|x(k^*)|^2 \\ &\quad + \gamma^{k^*+1}\sum_{i=1}^{s}\left|A^THB_i\right|\left[|x(k^*)|^2+|x(k^*-m_i)|^2\right] \\ &\quad + \frac{1}{2}\gamma^{k^*+1}\sum_{i,j=1}^{s}\left|B_i^THB_j\right| \\ &\quad \cdot\left[|x(k^*-m_i)|^2+\left|x\left(k^*-m_j\right)\right|^2\right]+\gamma^{k^*}(\gamma-1) \\ &\quad \cdot\lambda_{\max}(H)|x(k^*)|^2.\end{aligned} \tag{17}$$

Now we apply inequality (13) to get

$$\begin{aligned}\Delta V(x(k^*),k^*) &\le -\gamma^{k^*+1}\lambda_{\min}(C)|x(k^*)|^2 \\ &\quad + \gamma^{k^*+1}\sum_{i=1}^{s}\left|A^THB_i\right| \\ &\quad \cdot\left[|x(k^*)|^2+\gamma^{m_i+1}\varphi(H)|x(k^*+1)|^2\right]+\frac{1}{2} \\ &\quad \cdot\gamma^{k^*+1}\sum_{i,j=1}^{s}\left|B_i^THB_j\right|\left[\gamma^{m_i+1}+\gamma^{m_j+1}\right]\varphi(H) \\ &\quad \cdot|x(k^*+1)|^2+\gamma^{k^*}(\gamma-1)\lambda_{\max}(H)|x(k^*)|^2\end{aligned} \tag{18}$$

and

$$\begin{aligned}\Delta V(x(k^*),k^*) &\le -\gamma^{k^*+1}\left[\lambda_{\min}(C)-\sum_{i=1}^{s}\left|A^THB_i\right|\right. \\ &\quad \left.-\frac{\gamma-1}{\gamma}\lambda_{\max}(H)\right]|x(k^*)|^2+\frac{1}{2}\gamma^{(k^*+2)}\varphi(H) \\ &\quad \cdot\left[2\sum_{i=1}^{s}\gamma^{m_i}\left|A^THB_i\right|\right. \\ &\quad \left.+\sum_{i,j=1}^{s}\left[\gamma^{m_i+1}+\gamma^{m_j+1}\right]\left|B_i^THB_j\right|\right]|x(k^*+1)|^2.\end{aligned} \tag{19}$$

Inequality

$$\lambda_{\min}(C)-\sum_{i=1}^{s}\left|A^THB_i\right|-\frac{\gamma-1}{\gamma}\lambda_{\max}(H)>0 \tag{20}$$

can be deduced from the assumption $L_3 \ge 0$. Therefore, utilizing (8),

$$\begin{aligned}\Delta V(x(k^*),k^*) &\le -\gamma\left[\lambda_{\min}(C)-\sum_{i=1}^{s}\left|A^THB_i\right|\right. \\ &\quad \left.-\frac{\gamma-1}{\gamma}\lambda_{\max}(H)\right]\frac{V(x(k^*),k^*)}{\lambda_{\max}(H)}+\frac{1}{2}\gamma\varphi(H) \\ &\quad \cdot\left[2\sum_{i=1}^{s}\gamma^{m_i}\left|A^THB_i\right|\right. \\ &\quad \left.+\sum_{i,j=1}^{s}\left[\gamma^{m_i+1}+\gamma^{m_j+1}\right]\left|B_i^THB_j\right|\right] \\ &\quad \cdot\frac{V(x(k^*+1),k^*+1)}{\lambda_{\min}(H)}.\end{aligned} \tag{21}$$

Since $\Delta V(x(k^*),k^*)=V(x(k^*+1),k^*+1)-V(x(k^*),k^*)$, we get

$$\begin{aligned}&\left[1-\frac{1}{2}\gamma\frac{\varphi(H)}{\lambda_{\min}(H)}\left[2\sum_{i=1}^{s}\gamma^{m_i}\left|A^THB_i\right|\right.\right. \\ &\quad \left.\left.+\sum_{i,j=1}^{s}\left[\gamma^{m_i+1}+\gamma^{m_j+1}\right]\left|B_i^THB_j\right|\right]\right]V(x(k^*+1), \\ &k^*+1)\le\left[1-\frac{\gamma}{\lambda_{\max}(H)}\left[\lambda_{\min}(C)-\sum_{i=1}^{s}\left|A^THB_i\right|\right.\right. \\ &\quad \left.\left.-\frac{\gamma-1}{\gamma}\lambda_{\max}(H)\right]\right]V(x(k^*),k^*).\end{aligned} \tag{22}$$

This inequality can be rewritten as

$$\begin{aligned}&\frac{L_2}{\lambda_{\min}(H)}V(x(k^*+1),k^*+1) \\ &\quad \le\frac{L_1}{\lambda_{\max}(H)}V(x(k^*),k^*)\end{aligned} \tag{23}$$

or as

$$V(x(k^*+1),k^*+1)\le\Theta\cdot V(x(k^*),k^*) \tag{24}$$

where

$$\Theta := \frac{L_1}{L_2\varphi(H)}>0. \tag{25}$$

Now we prove that

$$\Theta\le 1. \tag{26}$$

Inequality (26) is equivalent with an inequality

$$\lambda_{\max}(H) - \gamma\left[\lambda_{\min}(C) - \sum_{i=1}^{s}\left|A^T H B_i\right| - \frac{\gamma-1}{\gamma} \cdot \lambda_{\max}(H)\right] \le \left[\lambda_{\min}(H) - \frac{1}{2}\gamma\varphi(H) \cdot \left[2\sum_{i=1}^{s}\gamma^{m_i}\left|A^T H B_i\right| + \sum_{i,j=1}^{s}\left[\gamma^{m_i+1} + \gamma^{m_j+1}\right]\left|B_i^T H B_j\right|\right]\right]\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}. \tag{27}$$

After some simplification, we get

$$\lambda_{\min}(C) - \sum_{i=1}^{s}\left|A^T H B_i\right| - \frac{\gamma-1}{\gamma}\lambda_{\max}(H) \ge \frac{1}{2}\varphi^2(H) \cdot \left[2\sum_{i=1}^{s}\gamma^{m_i}\left|A^T H B_i\right| + \sum_{i,j=1}^{s}\left[\gamma^{m_i+1} + \gamma^{m_j+1}\right]\left|B_i^T H B_j\right|\right], \tag{28}$$

which is equivalent with the inequality $L_3 \ge 0$. Then (24), (26), and (10) imply

$$\begin{aligned} V\left(x\left(k^*+1\right), k^*+1\right) &\le \Theta \cdot V\left(x\left(k^*\right), k^*\right) \\ &\le V\left(x\left(k^*\right), k^*\right) \le \varepsilon^2 \lambda_{\min}(H). \end{aligned} \tag{29}$$

This inequality contradicts (11). Then, inequality (11) is impossible and (10) holds for every $k = 1, 2, \ldots$. Moreover, (8) and (10) imply

$$\begin{aligned} \gamma^k \lambda_{\min}(H)\left|x(k)\right|^2 &\le V(x(k), k) \le \varepsilon^2 \lambda_{\min}(H) \\ &= \delta^2 \lambda_{\max}(H) \\ &= \left\|x(0)\right\|_m^2 \lambda_{\max}(H), \end{aligned} \tag{30}$$

i.e., the inequality

$$\gamma^k \lambda_{\min}(H)\left|x(k)\right|^2 \le \left\|x(0)\right\|_m^2 \lambda_{\max}(H), \quad k \ge 1, \tag{31}$$

equivalent with (7). □

3. Concluding Remarks

Based on the investigations on exponential stability published previously, the present paper brings in Theorem 1 new results. The exponential rate of convergence of solutions is studied in [1] assuming that $\det A \neq 0$; therefore, the results are independent. Let us discuss the independence of the results of other sources listed in the references. The criteria for the exponential stability of nonlinear difference systems, for example, are proved in [11, 14]. The nonlinearities are estimated by some linear terms with matrices having nonnegative entries with the sums of such matrices being, for example, a constant nonnegative matrix with a spectrum less than 1. In general, an attempt to estimate the right-hand sides of the systems by a nonnegative matrix does not provide a matrix with a spectrum less than 1 and the results are independent. For special classes of equations, sharp criteria (depending on delay) for detecting asymptotic stability are proved in [2, 3]. The following example illustrates the above-mentioned independency of results.

Example 2. Let $n = s = 2$ and let system (1) be of the form

$$x_1(k+1) = x_1(k) + x_2(k) + \mu x_2\left(k - m_1\right), \tag{32}$$

$$x_2(k+1) = -x_1(k) - x_2(k) + \nu x_1\left(k - m_2\right) \tag{33}$$

where $k \ge 0$ and μ and ν are constants. We show that Theorem 1 is applicable if $|\mu|$ and $|\nu|$ are sufficiently small. We have

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ \nu & 0 \end{pmatrix}. \tag{34}$$

Lyapunov equation (5) is satisfied, e.g., for

$$C = \begin{pmatrix} 0.9 & 0.9 \\ 0.9 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 1 \\ 1 & 1.1 \end{pmatrix}. \tag{35}$$

Then, $\lambda_{\max}(H) \doteq 2.0512492$, $\lambda_{\min}(H) \doteq 0.0487508$, $\lambda_{\min}(C) \doteq 0.0486122$, and $\varphi(H) \doteq 42.0762336$. Simple computations result in

$$\begin{aligned} \left|A^T H B_1\right| &= 0, \\ \left|A^T H B_2\right| &= 0.1\sqrt{2}\nu, \\ \left|B_1^T H B_1\right| &= \mu^2, \\ \left|B_2^T H B_2\right| &= 1.1\nu^2, \\ \left|B_1^T H B_2\right| &= \mu\nu, \end{aligned}$$

$$L_1 = \gamma \left[\lambda_{\max}(H) - \lambda_{\min}(C) + \sum_{j=1}^{2} \left| A^T H B_j \right| \right] \doteq \gamma \left[2.0026370 + 0.1\sqrt{2}\nu \right],$$

$$L_2 = \lambda_{\min}(H) - \frac{1}{2}\gamma\varphi(H) \left[2\sum_{i=1}^{2} \gamma^{m_i} \left| A^T H B_i \right| + \sum_{i,j=1}^{2} \left[\gamma^{m_i+1} + \gamma^{m_j+1} \right] \left| B_i^T H B_j \right| \right] \doteq 0.0487508 - \gamma 42.0762336 \left[\gamma^{m_2} 0.1\sqrt{2}\nu + \gamma^{m_1+1}\mu^2 + \left(\gamma^{m_1+1} + \gamma^{m_2+1} \right) \mu\nu + \gamma^{m_2+1} 1.1\nu^2 \right] \tag{36}$$

and

$$L_3 = \lambda_{\min}(C) - \sum_{i=1}^{2} \left| A^T H B_i \right| - \frac{\gamma - 1}{\gamma} \lambda_{\max}(H) - \frac{1}{2} \cdot \varphi^2(H) \left[2\sum_{i=1}^{2} \gamma^{m_i} \left| A^T H B_i \right| + \sum_{i,j=1}^{2} \left[\gamma^{m_i+1} + \gamma^{m_j+1} \right] \left| B_i^T H B_j \right| \right] \doteq 0.0486122 - 0.1\sqrt{2}\nu - \frac{\gamma - 1}{\gamma} 2.0512492 - (42.0762336)^2 \cdot \left[\gamma^{m_2} 0.1\sqrt{2}\nu + \gamma^{m_1+1}\mu^2 + \left(\gamma^{m_1+1} + \gamma^{m_2+1} \right)\mu\nu + \gamma^{m_2+1} 1.1\nu^2 \right]. \tag{37}$$

Theorem 1 is applicable if $|\mu|$ and $|\nu|$ are sufficiently small since this implies $L_i > 0$, $i = 1, 2$, and, if the expression

$$0.0486122 - \frac{\gamma - 1}{\gamma} 2.0512492 = \frac{2.0512492}{\gamma} - 2.0026370 \tag{38}$$

is positive, provided that $\gamma > 1$; that is, if

$$1 < \gamma < \frac{2.0512492}{2.0026370} \doteq 1.0242741, \tag{39}$$

then $L_3 > 0$ as well. In such a case, for an arbitrary solution $x(k) = (x_1(k), x_2(k))^T$ of system (32), (33), the estimate

$$|x(k)| \leq \sqrt{\varphi(H)} \, \|x(0)\|_m \, \gamma^{-k/2} \doteq 42.0762336 \, \|x(0)\|_m \, \gamma^{-k/2}, \quad k \geq 1 \tag{40}$$

holds.

Since $\det A = 0$ in the above example, the results of the paper [1] are not applicable to system (32), (33). Moreover, an attempt to apply results of [11, 14] is not successful since the sum of matrices A^*, B_1^*, and B_2^*, defined by replacing the entries in the previously given matrices A, B_1, and B_2 by their absolute values, leads to a matrix

$$U := A^* + B_1^* + B_2^* = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & |\mu| \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ |\nu| & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 + |\mu| \\ 1 + |\nu| & 1 \end{pmatrix} \tag{41}$$

whose eigenvalues are $\lambda_{1,2}(U) = 1 \pm \sqrt{(1 + |\mu|)(1 + |\nu|)}$, and, obviously, $\rho(U) \geq 1$.

Finally, we compare the results published in [4–7] with Theorem 1. The assumptions of Theorem 1 are, for the reduced case $s = 1$ of a single delay, weaker than those of Theorem 2 in [7]. In [4] an analysis of Theorem 2 is carried out. Although the results are independent, a limiting process (for $\gamma \longrightarrow 1^+$) indicates that the conditions of the main result in [7] are, in general, more restrictive. Now we will demonstrate that, with respect to the derived estimates of the norms of solutions, the situation is just the opposite and that the estimation (7) is, in general, better than that in [4, Theorem 2]. The last estimation mentioned says that (below, s, A, B_i, $i = 1 \ldots, s$, H and C are the same as in the paper)

$$|x(k)| \leq \sqrt{\varphi(H)} \, \|x(0)\|_m \, \Theta^{k/2(m+1)}(H), \quad k \geq 1, \tag{42}$$

where

$$\Theta(H) := \frac{1}{\lambda_{\max}(H)} \left[L(H) - \sum_{i=1}^{s} L_i(H) + s\lambda_{\min}(H) \right],$$
$$L(H) := \lambda_{\max}(H) - \lambda_{\min}(C) + \sum_{j=1}^{s} \left| A^T H B_j \right|,$$
$$L_i(H) := \lambda_{\min}(H) - \varphi(H) \left[\left| A^T H B_i \right| + \sum_{j=1}^{s} \left| B_i^T H B_j \right| \right], \quad i = 1, \ldots, s, \tag{43}$$

if $\rho(A) < 1$, C is a fixed positive definite matrix, matrix H solves the corresponding Lyapunov matrix equation (5), and

$$L(H) - \sum_{i=1}^{s} L_i(H) < \lambda_{\max}(H) - s\lambda_{\min}(H), \quad L(H) > 0. \tag{44}$$

Assuming that $|B_i| \longrightarrow 0$, $i = 1, \ldots, n$, we deduce that for (44) to hold, the following is necessary:

$$\lambda_{\max}(H) - \lambda_{\min}(C) > 0, \tag{45}$$

the limiting value of $\Theta(H)$ is

$$\Theta(H) \doteq \frac{\lambda_{\max}(H) - \lambda_{\min}(C)}{\lambda_{\max}(H)}, \tag{46}$$

and (42) can approximately be written as

$$|x(k)| \leq \sqrt{\varphi(H)}\, \|x(0)\|_m \left[\frac{\lambda_{\max}(H) - \lambda_{\min}(C)}{\lambda_{\max}(H)}\right]^{k/(2(m+1))}, \quad k \geq 1. \tag{47}$$

Considering the same limiting process as above, for the validity of (7), an analysis of L_i, $i = 1, 2, 3$ implies that inequality (45) must hold in addition to inequality

$$\lambda_{\min}(C) - \lambda_{\max}(H) + \frac{1}{\gamma}\lambda_{\max}(H) > 0, \tag{48}$$

derived from the assumption $L_3 \geq 0$. Inequality (48), together with the assumption $\gamma > 1$, yields

$$1 < \gamma < \frac{\lambda_{\max}(H)}{\lambda_{\max}(H) - \lambda_{\min}(C)} \tag{49}$$

and (7) can be approximatively written as

$$|x(k)| \leq \sqrt{\varphi(H)}\, \|x(0)\|_m \left[\frac{\lambda_{\max}(H) - \lambda_{\min}(C)}{\lambda_{\max}(H)}\right]^{k/2}, \quad k \geq 1. \tag{50}$$

Obviously, estimation (50) is (due to the absence of the maximal delay m) better than estimation (47). We finish this part with a remark that the results of [5] are generalized in [4]. Results of [6] are on the exponential stability of linear perturbed systems with a single delay. Among others, it is proved [6, Theorem 3] that inequality (50) holds for nondelayed linear systems

$$x(k+1) = Ax(k), \quad k = 0, 1,, \ldots. \tag{51}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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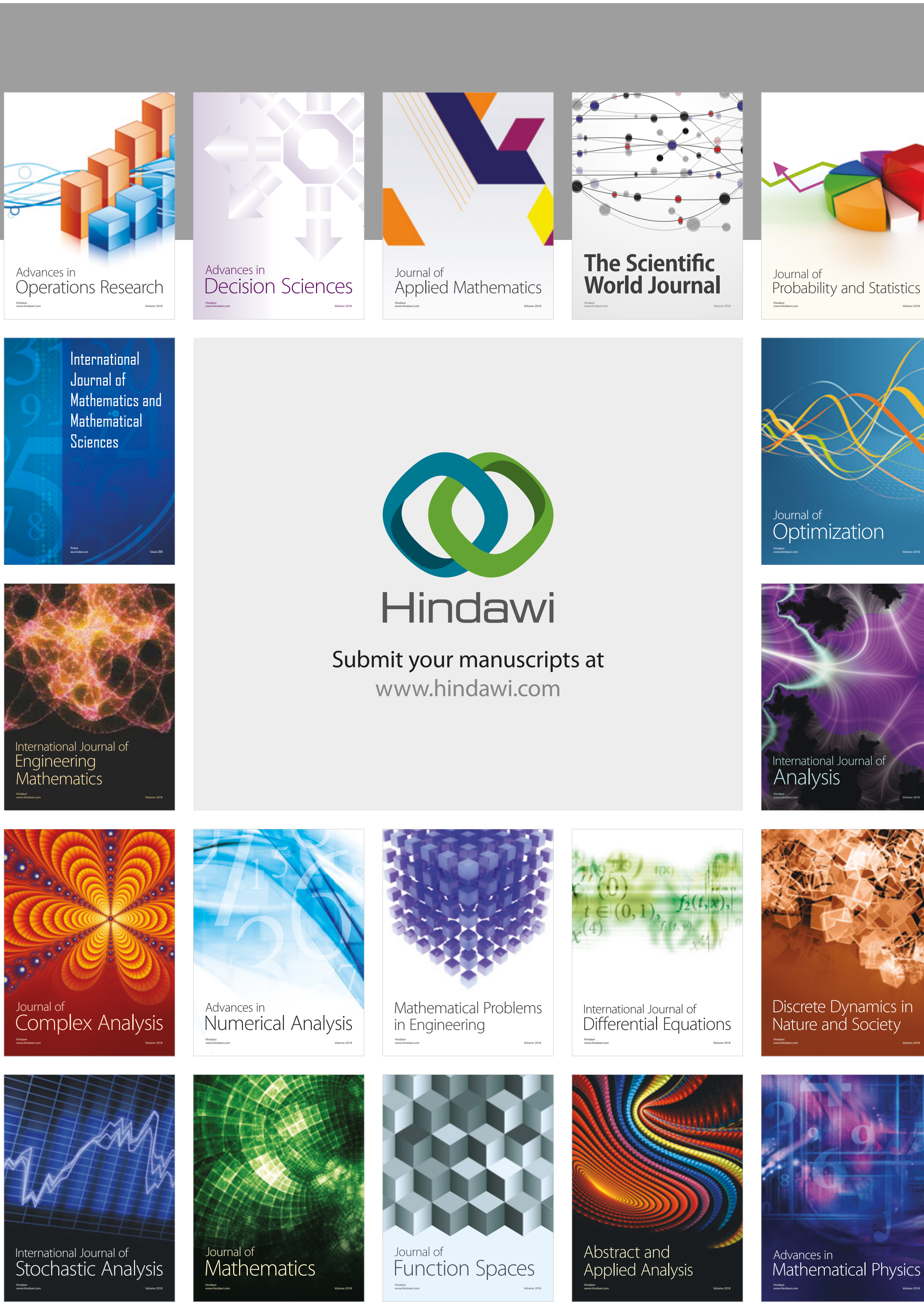

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